DESIGN OF ANISOTROPIC ANNULAR PLATES WITH THE USE OF NUMERICAL METHODS

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Two problems of optimal design of concentric annular plates are considered. The plates are made from an anisotropic composite material and are loaded by a uniform pressure p either along the inner or outer contour. Using the procedure of setting up rational designs [1, 2], solutions are written down which are not optimal but are subsequently numerically improved.

The material is assumed to consist of a bonding matrix reinforced by thin high-strength fibers. When setting up the basic equations, it is considered as a solid anisotropic medium. The fibers are oriented in radial (1) and circumferential (2) directions. The volume intensities of the reinforcement have the same values s_1 and s_2. These are nonnegative quantities whose sum must not exceed a certain value s^* determined from technological considerations.

It is assumed that the radial stresses σ_1 and the circumferential stresses σ_2 are connected with the corresponding strains ε_1 and ε_2 and the intensity of reinforcement by the linear relationship

$$\sigma_i = E s_i \varepsilon_i \qquad (i = 1, 2) \tag{1}$$

where E is a constant of the material.

The strains are bounded by a certain limiting value ε^*.

For the radii a, b (a < b) of the plate and a constant thickness H the volume of reinforcement is written in the form

$$V = 2\pi H \int_a^b (s_1 + s_2)\, r\, dr \tag{2}$$

In order to make the solutions suitable for materials with any values of the constants E, s^*, ε^*, we introduce the dimensionless quantities

$$\begin{gathered} e_i = \varepsilon_i / \varepsilon^*, \quad \zeta_i = s_i / 2s^*, \quad \tau_i = \sigma_i / 2Es^*\varepsilon^* \\ q = p / 2Es^*\varepsilon^*, \quad W = V / \pi H a^2 s^* \end{gathered} \tag{3}$$

which retain all the names of the corresponding quantities with dimensions.

The variables e_i, τ_i, ζ_i are functions of the dimensionless radius $\rho = r/a$.

According to the law (1) the intensities of reinforcement can be expressed in terms of stresses and strains $\zeta_i = \tau_i / e_i$. The basic quantities τ_1, τ_2, e_1, e_2 satisfy the conditions of equilibrium and compatibility

$$\tau_2 = \rho\, d\tau_1/d\rho + \tau_1, \qquad e_1 = \rho\, de_2/d\rho + e_2 \tag{4}$$

and constraints in the form of inequalities determined by the requirements of positiveness of reinforcement intensity and boundedness of strains and the overall reinforcement intensity,

$$\tau_i / e_i \geqslant 0, \quad e_i \leqslant 1, \quad \tau_1 / e_1 + \tau_2 / e_2 \leqslant 2 \tag{5}$$

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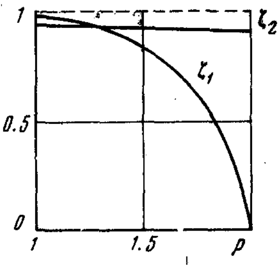

Fig. 1

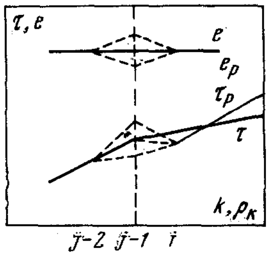

Fig. 2

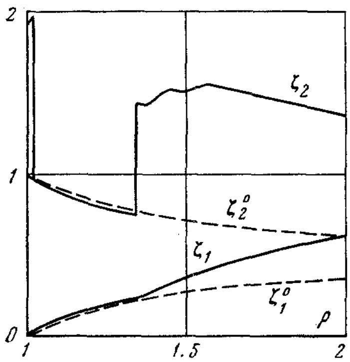

Fig. 3

Problem 1. A uniform pressure q acts on the inner controur (ρ = 1); the outer contour (ρ = 2) is free. A plate having the least volume of reinforcement and supporting a given load is to be designed.

As is shown in [2], using the solution of the problem of the theory of elasticity for an isotropic annulus, we can set up a rational design of an anisotropic body given by the following functions, which here will be taken as the initial approximation:

$$\tau_1^\circ = e_1^\circ = -0.8/\rho^2 + 0.2, \qquad \zeta_1^\circ \equiv 1$$
$$\tau_2^\circ = e_2^\circ = 0.8/\rho^2 + 0.2, \qquad \zeta_2^\circ \equiv 1 \tag{6}$$

The procedure of obtaining these functions is inessential for the present discussion, since we can easily convince ourselves immediately that (6) indeed satisfies the relationships (4) and (5). The boundary values of the function τ_1° also agree with the given method of loading the annulus: pressure is absent on the outer contour and $\tau_1^\circ(2) = 0$; a pressure q is given on the inner contour and $\tau_1(1) = -0.6$. Thus, for a pressure q = 0.6 the function (6) determines a rational design which will be improved by a numerical method, so that for the unaltered value of q we can find the new functions τ_1, τ_2, e_1, e_2 and ζ_1, ζ_2 expressed in terms of them, which give a smaller value of the functional (4).

For the initial approximation (6) the relative volume of reinforcement $W^\circ = 3$.

The solution of the problem was carried out by the method of local variations [3]. The interval [1, 2] of the variable is divided into 100 equal parts, continuous functions are replaced by discrete ones, and the differential relationships are replaced by finite difference relationships. In the first variation cycle the functions received an increment h = 0.01, while in the last cycle they received h = 0.0004. The last but one variation cycle with a step of h = 0.00167 gave the value of W different by only 0.5% from the preceding final value equal to W = 2.427. The economy in the volume of reinforcement, in comparison with the initial design, amounts to 19%. The program was produced in the ALGEM language, and the computation was carried out on a Minsk-22 computer.

In Fig. 1 we have presented the functions ζ_1, ζ_2, which define the reinforcement intensities in the radial and circumferential directions for the final design (the dashed line is the initial graph of these functions according to (6)).

Problem 2. A uniform pressure q acts on the outer contour (ρ = 2) of the plate; the inner contour (ρ = 1) is free. A plate is to be designed for which the parameter q assumes a maximum value, i.e., a plate which is optimal with respect to strength. The volume of reinforcement must not exceed the maximum possible value, equal to 3.

We know of a design which is optimal with respect to the mass (see [2]) and which, for the dimensionless pressure $q^\circ = 0.375$, has the volume of reinforcement $W^\circ = 1.5$ and is characterized by the following functions:

$$\tau_1^\circ = 0.5/\rho^2 - 0.5 = -\zeta_1^\circ, \qquad e_1^\circ \equiv -1$$
$$\tau_2^\circ = -0.5/\rho^2 - 0.5 = -\zeta_2^\circ, \qquad e_2^\circ \equiv -1 \tag{7}$$

It is easy to see that these functions satisfy the relationships (4), (5) and the boundary conditions of the given problem. Proceeding from the functions (7) as from the initial approximation, we must vary them in such a way that the functional $q = -\tau_1(2)$ would assume the largest value. This functional, defined on the functions τ_1 and e_2 (the rest of them are expressed in terms of these two, as can be seen from (4) and

(5)), is not additive. In view of this, the method of local variations in the given case is not applicable, and the following algorithm of improving the functional is used to solve the problem.

The interval of variation of the independent variable [1, 2] is divided into 100 different parts with the points ρ_k $(0 \le k \le 100)$. Instead of continuous functions we introduce discrete functions; for example, instead of the basic functions $\tau_1(\rho)$ and $e_2(\rho)$ we consider the functions $\tau[k]$, $e[k]$, where $\tau[k] = -\tau_1(\rho_k)$, $e[k] = -e_2(\rho_k)$ (the minus sign is introduced to make positive the functions subject to variation). The derivatives of these functions are replaced by the "left" ratios of finite differences; for example, $d\tau_1/d\rho$ for $\rho = \rho_k$ is replaced by $-(\tau[k] - \tau[k-1])/100$. The relationships (4), (5) and the expression for the reinforcement volume are rewritten in such a way that they contain only the functions $\tau[k]$, $e[k]$ whose initial approximations τ°, e° are found from (7). Starting from these functions, we construct the first tentative approximation $\tau_p{}^1$, giving the value of the function τ, for k = 100, the increment

$$\tau_p{}^1[k] = \tau^\circ[k] \quad (k = 0, 1, 2, \ldots, 99), \qquad \tau_p{}^1[100] = \tau^\circ[100] + h$$

If this function satisfies all constraints R (given by the conditions (4), (5), the boundary conditions, and the constraint on the reinforcement volume $W \le 3$), then we consider it the first true approximation τ^1. Each new tentative approximation is started to be set up from the fact that $\tau^i[100]$ is given a positive increment. In view of this the functional $q = \tau[100] = \tau_1(\rho_{100})$ in such a process cannot decrease.

In Fig. 2 with thick lines we have provisionally depicted the true approximation for the functions, while with thin lines we have shown a tentative approximation. The process of constructing the latter went as far as the point $k = j-1$. With dashed lines we have shown possible variants of setting up tentative approximations at the point $j-1$. If for any choice of values at the point $j-1$ the constraints R on the tentative functions, which are being tested at the point j, are not fulfilled, then all tentative approximations (thin lines) are discarded and the variation process is started from the beginning (with a smaller value of the step h), proceeding from the last true approximation (thick lines). If, however, at the point j the conditions R are fulfilled, they are verified at the point $j-1$. If they are satisfied at this point, the tentative approximation is considered to be true, since on the left of this point the functions and their derivatives were not varied; otherwise, we go over to the point $j-2$.

The sorting out of variants produced in this manner does not ensure the comparison of all functions which are close to the given true approximation, so that we cannot say that the process just described necessarily leads to the maximum value of the functional. However, it leads to an improvement in the functional. In practice this improvement can be very substantial.

A program was set up in ALGOL according to this algorithm and the computation was carried out on a BÉSM-4 computer at the Computer Center of Moscow State University. The final graphs of the functions ζ_1 and ζ_2 are shown in Fig. 3. The discontinuity of the function ζ_2 does not contradict the nature of the problem, since the reinforcement intensity in the circumferential direction can indeed vary with a jump. The functions e_1 and e_2 varied only slightly; here we virtually can consider $e_1 \equiv -1$, so that the graph for ζ_1 gives a full picture of the final variant of the basic function τ_1 subject to variation. The initial form of this function is shown with a dashed line $(\tau_1{}^\circ \equiv -\zeta_1{}^\circ)$.

The variation step was varied from 0.01 to $0.61 \cdot 10^{-6}$. The largest value of the functional was q = 0.621; the reinforcement volume increased up to W = 2.57.

It can be shown that the problem considered here has an exact solution yielding an absolutly optimal design

$$\tau_1{}^* = 1/\rho^2 - 1 = -\zeta_1{}^*, \qquad e_1{}^* \equiv -1$$
$$\tau_2{}^* = -1/\rho^2 - 1 = -\zeta_2{}^*, \qquad e_2{}^* \equiv -1$$

The reinforcement volume of such a design equals the limiting possible value $W^* = 3$; the strength is $q^* = 0.75$. In Fig. 3 the dashed lines depict the dependence for the continuous functions $0.5\zeta_1{}^*$, $0.5\zeta_2{}^*$. Comparing them with the numerical solution obtained, we can conclude that it leads to a discontinuous function ζ_2 which is very much different from $\zeta_2{}^*$.

The quantitative results of the numerical solution must be regarded as satisfactory; the specific strength q/W of the design set up by the numerical method is roughly equal to the specific strength of the optimal design. The relative shortage with respect to the strength amounts to 17%; with respect to the reinforcement volume it amounts to 14%. In comparison with the initial approximation the strength increased by 65%.

LITERATURE CITED

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